Eleven Great Problems of Mathematical Hydrodynamics

V. I. Yudovich

The prince in person led his troops to attack eleven times. Alexander Dumas.

Eleven times the foolhardy battalion attacked the enemy. Nikolay Tikhonov.

1 Introduction

Regardless of the fact that this paper was written on the other occasion, I respectfully dedicate it to Vladimir Arnold. An interviewer once asked me whether I have heroes in mathematics. I said that I have just heroes ,not gods, and first mentioned Arnold. His unique ability to respond to all alive and new in mathematics and physics by unexpected and stimulating ideas, his impeccable mathematical taste, his extraordinary penetrating power, making us to recall classics, his remarkable gift to point out the research directions promising maximal results — all this made him one of the world leaders in modern mathematics.

This is a slightly extended version of a talk that was given at the Conference on mathematical hydrodynamics at Hull University, UK, on the 10th April 2001. This talk was also repeated at the Newton Institute, Cambridge, on the 23rd April 2001. The title of this paper was suggested by V. A. Vladimirov, who invited me to participate at the conference in Hull. "Why exactly eleven ?", I asked him. "For no particular reason", he replied. "Hilbert pointed out 20 problems, and Smale pointed out 19, but these referred to mathematics in general... Well, if you do not like it, we can change the title." However, I liked the number 11. It is not as common as, for instance, 12 or 7. Besides, I recalled that some of my favorite authors mention this number while talking about battles and attacks. During the talk, I was reminded that a football team consists of 11 players. More recently, I read in Kazantsev's article in *Shahmatnoe Obozrenie* (Chess Review) 2, 2001, that "11, according to investigations of V. I. Avinsky, a founder of alphametrics, is a module of the Universe involved in all dimensions of both micro- and macro-worlds." So, this is an intriguing number and let it stay in the title, although one shouldn't take it too seriously. As a matter of fact, there is actually a greater number of problems and almost all of them split into even more new problems. While choosing the 11 problems for this list I tried to use the following criteria:

1. The solution of the problem should bring us to a new level of understanding of fluid dynamics, or at least help to explain a sufficiently wide range of hydrodynamical phenomena.

2. In general, it is impossible to solve the problem with any known method. We need some new ideas and approaches which will probably give rise to new mathematical theories and promote the improving art of the description of natural phenomena.

3. Again quoting S. Smale [1], "We believe that the questions, their solutions, partial results, or even attempts to solve them are likely to be of great importance for mathematics and its development in the next century."

4. I spent a good amount of time trying to solve the problem and know some things about it. At the very least, I now know of several approaches that do not lead to the desirable results.

As a result almost all of the problems in this list deal with incompressible homogeneous ideal and viscous fluids. However, some others were excluded from the list only for the sake of brevity. Those were different problems concerning compressible fluids, nonhomogeneous fluids, asymptotic models of convection, magnetohydrodynamics, multi-component and especially infinite-component media, analytical dynamics and differential geometry of continuous media, different problems with the unknown and particularly free boundaries, etc. Some of those omitted still satisfy all of the above four criteria, and I hope to return to them in future publications.

Of course, it would be desirable that a problem be formulated in a mathematically rigorous manner. However, this is unfortunately not always possible. I suppose that a physical problem cannot be formulated completely until it is resolved. Only a beautiful solution will eventually confirm the correctness of the problem's initial statement.

Many of the problems discussed below are well-known, while some of them are rather new. In the current mathematical literature, one can find solutions for probably all known problems, especially for those where the hypothetic result is sufficiently clear and needs only to be rigorously justified. According to Francois Rabelais, "every respectable citizen should believe everything he is told and everything which is published." In mathematical hydrodynamics, this great principle should be applied carefully since too many published proofs are erroneous.

The following is a list of problems with some comments. The first two problems concern the fundamentals of mathematical physics and are not part of the list of eleven great problems in fluid dynamics.

2 Mathematical Models of Hydrodynamics.

Problem G1. *Construct mathematical models of continuous media including phase transitions (boiling water, ferroelectrics which can turn into dielectrics, liquid crystals, etc.).*

This is mainly a question of the correct mathematical statement of the initial boundary-value problem under conditions when a continuous medium can undergo phase transitions in unknown *a priori* moments and in unknown *a priori* regions of space occupied by it. For example, it is necessary to learn how to describe the flow of water under conditions when its temperature changes within the interval containing one or several points of phase transition (freezing — melting, boiling — condensation, or triple critical point of the equation of state nearby which all three phases can coexist). This problem belongs as much to physics as to mathematics, since the interpretation of the phase transitions is still an unsettled area of physics. Current physical journals regularly publish works on the fundamentals of this theory (see for example [2]).

Available phenomenological models of liquid-gas mixture and boiling water are rather rough, while it would be interesting to obtain the appropriate equations starting from the "first principles" of statistical thermodynamics. By the way, the possibility of transition of water into ice reminds us once again about the impossibility to establish the partitions between the natural sciences once and for all, since such partitions do not exist in nature.

Interfaces arising at phase transitions very often turn out to be unstable and waves appear on them. A number of interesting problems are connected with these phenomena; many of them do not even require the creation of new methods and are quite accessible to investigation. I recall the experiment conducted by Rostov physicists (Fridkin and Grekov) [3] as long ago as the 1970s. The edges of a rod made of ferroelectric material (such as barium titanate) were kept at constant temperatures. The temperature on one edge was lower than the Curie point and on the other was higher. One could expect that the part of the rod close to the hot edge would be in dielectric phase and the cold part would be in ferroelectric phase. In general the experiment confirmed these expectations, but the point of interface started to oscillate along the rod. As far as I know, appropriate theory of this phenomenon was never constructed.

Another example: it is doubtless that a flat boundary between water and ice while freezing or melting is often unstable. It would be of interest to investigate spontaneous waves appearing on such a boundary. It would also be of great interest to consider parametrically excited waves generated by oscillations of outer temperature and pressure. The results may become significant for the investigation of glacier motions, formation and thawing of icebergs, and freezing of water reservoirs. The obtained results can also be applied in the development of practical methods for the destruction of ice covers on rivers and lakes.

Problem G2. *Determine the dependence of kinetic coefficients (viscosity, thermoconductivity, diffusion, surface tension, permittivity, ...) on thermodynamic parameters (temperature T, pressure p, density ρ, impurity concentration c, ...).*

It is very important to determine the restrictions on kinetic coefficients imposed by the requirement of global solvability in basic evolution initial boundary-value problems. I believe that it is possible to build up the general theory of globally solvable systems of ordinary differential equations and equations in partial derivatives. Of course, global existence of solutions to initial boundary value problems is a very special *physical* property of the system, since, as we know, there are some explosive continuous media which can exist only within a limited period of time. Speaking in mathematical language, the possibility of collapse is a generic property, while the global solvability is in a sense a degeneration. (Such is the eccentric mathematical language — the most interesting and beautiful systems are called *degenerate.*)

3 Uniqueness, global existence and nonexistence of a solution.

Problem 1. *Global solvability and regularity of the solutions to the basic boundary-value problems for* $3D$ *Euler and Navier-Stokes equations in the case of homogeneous incompressible fluid.*

There is no point in giving a detailed description of these well-known problems even more so that recently I wrote an extensive article about them [4]. However, it is worth noting that similar problems arise in many areas of nonlinear mathematical physics. The situation in the $2D$ Euler and Navier-Stokes equations is good enough since global theorems on the existence of generalized and smooth solutions are known as well as rather strong uniqueness theorems. That is why it is widely believed that only $3D$ problems are difficult while $2D$ problems are not so hard to solve. As a matter of fact, "$2D$ or not $2D$, that is not the question". The issue here is not so much two-dimensionality but the specific properties of the Euler and Navier-Stokes equations, which make it possible to obtain strong *a priori* estimates of the solutions. In the case of Euler equations this is the existence of the vortex integrals. In the case of Navier-Stokes equations this is the specific embedding theorems for the function spaces that play the decisive role. Kinetic energy in the $3D$ case is still a quadratic functional; however in order to follow in a fashion similar to that for plane flows, one needs the velocity norm in L_3 space (in the n-dimensional case we need L_n).

If we consider the generalized solutions of Euler equations with initial velocity fields, which possess only finite kinetic energy, with no additional assumptions on smoothness, then the advantage of the $2D$ case fails immediately. The question of global solvability and uniqueness of the solution to the basic initial boundary-value problem in the $2D$ case turns out to be as complicated as for the similar $3D$ problem.

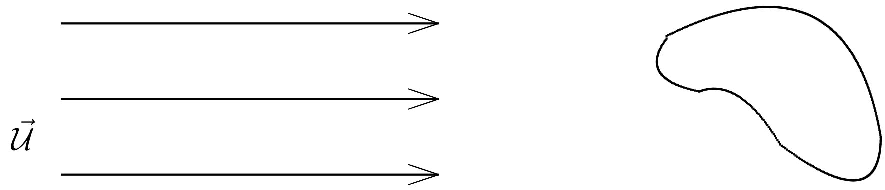

Figure 1:

Let us consider also the equations of ideal convection:

$$\frac{d\vec{v}}{dt} = -\vec{\nabla}p + \theta\hat{k} \quad (\text{where } \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}), \tag{1}$$

$$\vec{\nabla}\cdot\vec{v} = 0, \tag{2}$$

$$\frac{d\theta}{dt} = 0, \tag{3}$$

$$v_n\big|_{\partial D} = 0,\ \vec{v}\big|_{t=0} = \vec{v}_0,\ \theta\big|_{t=0} = \theta_0, \tag{4}$$

where $\vec{v}$ is the velocity field, p is the pressure, θ is temperature, $\hat{k}$ is a unit vector directed upwards, D is a bounded domain in $\mathbb{R}^3$ or $\mathbb{R}^2$; $\vec{v}_0 \equiv \vec{v}_0(x)$, $\theta_0 \equiv \theta_0(x)$, $x \in D$, are initial velocity and initial temperature fields.

If $\theta_0 = const$, then $\theta(x,t)$ is constant for all x and t, and we face the problem for Euler equations. Proof of the global existence theorem in the class of smooth solutions is quite inaccessible for non-isothermal flows even in the $2D$ case. The question of whether or not blow up is possible arises once again. Two-dimensionality is of no use in this case, since the conservation law for the vorticity in a fluid particle is no longer valid.

In the right way all these problems are stated informally: one should find the proper definition of the (generalized) solution so that both the global existence theorem and the uniqueness theorem can be proved.

Problem 2. *Global existence theorems for stationary and periodic flows.*

After the classic works by J. Leray [7] and his successors (see, for instance, [5]) the following two problems still resist the efforts of researchers.

Problem 2a. *Global existence theorem for the solution to the $2D$ problem on the viscous fluid flow past a rigid body.*

The velocity at infinity is assumed to be given and equal to the prescribed constant vector $\vec{U}$ (see Fig.1).

This problem goes back to the Stokes paradox. Stokes established that in the linear case with the neglected term $(v, \nabla)v$ the solution doesn't exist. This is in a sharp contrast with the existence of a $3D$ flow past a bounded body, which result is very important from a practical point of view. The Stokes solution for the slow flow past a sphere has numerous applications in the natural sciences.

We could reverse the problem and consider translational movement of the infinite rigid cylinder with a constant velocity $-\vec{U}$ in the fluid which is at the rest at infinity. Stokes' result implies that in the course of time (as $t \to \infty$) all the fluid will start to move with the same velocity $-\vec{U}$ and the condition at infinity will be violated. The question is whether this result will change if we consider the complete Navier-Stokes equations. It is worth to notice that all the principal results for the Navier-Stokes equations were obtained, so to say, *in spite* of nonlinearity. The results that we can get more or less easily for linearized equations afterwards were extended (while struggling against nonlinearity!) to the complete equations. As for the $2D$ flow problem the desired result must be obtained *with the help* of nonlinearity. So far this has been done only for low Reynolds numbers [6].

Similar questions for non-translational motions of a body and for motions periodic in time still stay without proper investigation.

Problem 2b. *Prove or disprove the global existence of stationary and periodic flows of a viscous incompressible fluid in the presence of the interior sources and sinks.*

Let us consider the following steady-state boundary value problem for the Navier-Stokes system. Let the flow domain D of $\mathbb{R}^3$ or $\mathbb{R}^2$ has a boundary ∂D consisting of connected components $S_1, S_2, \ldots, S_k$. Let the velocity v be prescribed along the boundary:

$$v|_{\partial D} \;=\; q, \tag{5}$$

where q is a given vector field on the boundary. Then the incompressibility condition (2) imposes certain restriction on the vector field q:

$$\sum_{l=1}^{k} \int_{S_l} q_n dS \;=\; 0, \tag{6}$$

(i. e. the total velocity flux through the boundary ∂D must be equal to zero). Meanwhile, in the classic work by J. Leray [7] the global existence theorem for the stationary flow was proved only for the more restrictive condition:

$$\int_{S_1} q_n dS = \ldots = \int_{S_k} q_n dS = 0, \tag{7}$$

which coincides with the necessary condition (6) only in the case of a connected boundary (i. e. $k = 1$). Condition (7) means that the fluid neither enters the flow domain D from the interior domains bounded by the surfaces S_1, $S_2, \ldots, S_k$ (we assume that S_k is the outer boundary of the flow domain), nor leaves the domain D through these surfaces. So there are no interior sources or sinks (more precisely, their sum is equal to zero).

It was necessary to impose the same boundary condition (7) on the boundary field $q(x, t)$ in order to prove the global existence theorem for periodic motions [8]. We face the following problem:

Prove (or disprove by constructing a counterexample) the global theorem on existence of stationary and forced periodic motions of viscous incompressible fluids in the case when the domain D in $\mathbb{R}^2$ or $\mathbb{R}^3$ has the boundary which consists of the connected components $S_1, \ldots, S_k$, where $k > 1$ and only the necessary condition (6) holds.

Here the boundary ∂D, the vector field q, and the external mass force $F(x)$ in the stationary case and $F(x, t)$ in the periodic case are considered to be C^∞-smooth.

My feeling is that the result most likely is negative. If this is true, then the necessary counter-examples can be constructed, probably even for the simplest case of the concentric circular ring.

The exterior problems with the interior sources and sinks are also not without interest. Some unusual phenomena related to the outer rotationally symmetric flows are considered in [9].

In addition, I would like to note that condition (7) makes it possible to prove the dissipativity of the non-stationary Navier-Stokes system [10]. When only the general condition (6) is valid, this result (except in the slow flow case) will probably fail as well.

It seems to be possible that while the Reynolds number increases, the stationary regime can disappear (i. e. move to infinity in the corresponding function space), when $R \to R_*$, where the critical value of R_* is finite.

However, before disappearing, this stationary regime becomes unstable and generates the self-oscillating periodic regime. Of course, there is a chance that at first the branching in the class of stationary regimes takes place. It would be interesting to examine the possibility of such kind march of events at least with a numerical experiment.

4 General stability theory for viscous fluid flows.

Problem 3. *Existence of unstable stationary and periodic flows in an arbitrary domain.*

Let $a = a(x)$ be a velocity field of a stationary flow of a viscous incompressible fluid in a prescribed bounded domain D in $\mathbb{R}^2$ or $\mathbb{R}^3$. We assume a to be a solution to the boundary-value problem for the Navier-Stokes system with prescribed external forces and boundary velocity field. By linearizing the Navier-Stokes equation on this *basic flow* and searching for a solution in the form $e^{\sigma t}u(x)$, we get the following spectral problem:

$$\sigma u + (u,\nabla)a + (a,\nabla)u = -\nabla q + \nu\Delta u, \tag{8}$$

$$\nabla \cdot u = 0, \tag{9}$$

$$u|_{\partial D} = 0. \tag{10}$$

We call the *stability spectrum* of the main flow a (denote by Σ_a) the set of the complex numbers σ for which the problem (8)–(10) has a nonzero solution. It is well-known that the stability spectrum of any flow is countable and the corresponding system of eigenvectors and adjoint vectors is complete [14].(Note that in the case of unbounded domain one should consider also the *continuous spectrum.*)

Let us try to imagine the great future hydrodynamic stability theory that has already solved all the fundamental problems and is able to entrust with computers the investigation of particular flows, their stability and transitions. May be the following concepts will play an essential role in this theory.

Definition 1. *Let us call destabilizer $\mathcal{D} = \mathcal{D}(D)$ the set of all smooth solenoidal (*$\operatorname{div} v = 0$*) vector fields a on the domain D such that the spectral problem (8) -(10) has at least one eigenvalue σ_0 on the imaginary axis:*

$$\mathcal{D} = \mathcal{D}(D) = \{a : \ \nabla \cdot a = 0, \ \exists \ \ \sigma_0 \in \Sigma_a : Re\ \sigma_0 = 0\}. \tag{11}$$

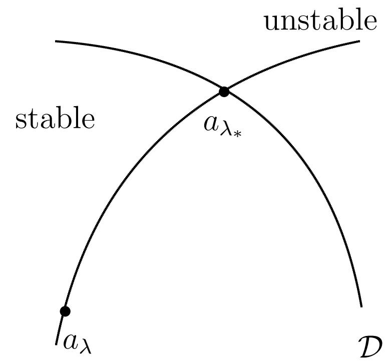

Figure 2:

Definition 2. *Let us call bifurcator $\mathcal{B} = \mathcal{B}(D)$ the set of all solenoidal vector fields a on the domain D such that the stability spectrum Σ_a contains the point 0:*

$$\mathcal{B} = \mathcal{B}(D) = \{a : \ \nabla \cdot a = 0, \ 0 \in \Sigma_a\}. \tag{12}$$

Definition 3. *Let us call oscillator $\mathcal{O} = \mathcal{O}(D)$ the set of all smooth solenoidal vector fields a on the domain D such that the stability spectrum Σ_a contains at least one pair of complex conjugate numbers $\pm i\omega$, $\omega \neq 0$:*

$$\mathcal{O} = \mathcal{O}(D) = \{a : \ \nabla \cdot a = 0, \ \exists \ \omega \in R, \ \omega \neq 0, \ i\omega \in \Sigma_a\}. \tag{13}$$

Now let $a_\lambda = a_\lambda(x)$ be a solenoidal vector field depending on the real parameter λ. Assume that if $\lambda = 0$, this flow $a_0 = a_0(x)$ is asymptotically stable, and its stability spectrum is situated in the left half-plane. Then the property of asymptotic stability is also preserved for small λ. Let us now gradually increase λ (without loss of generality we can assume $\lambda > 0$). It is possible that the flow a_λ will be unstable for some λ. The critical values of λ_*, which correspond to a transition of the eigenvalues from the stable half-plane to the unstable one (particularly those which separate the intervals of stability and instability), are determined by the condition that the spectrum Σ_a includes at least one point of the imaginary axis. In other words, the critical values λ_* are defined by the condition: $a_{\lambda_*} \in \mathcal{D}(D)$ (Fig.2). Of course while we change the parameter λ, the curve $\{a_\lambda\}$ can cross the destabilizer $\mathcal{D}$ several times.

If it is already known that the flow a_λ loses its stability, the question arises about the nature of the corresponding transition. Generically, the answer depends mainly on the nature of the *neutral spectrum* (intersection of the spectrum with the imaginary axis) of the critical flow a_{λ_*}. If $a_{\lambda_*} \in \mathcal{B}(D)$, one can expect the branching of the stationary regimes. And if $a_{\lambda_*} \in \mathcal{O}(D)$, then (again generically) the Poincare-Andronov-Hopf bifurcation of branching off the cycle (self-oscillatory periodic regime) takes place.

If we imagine that the sets $\mathcal{D}, \mathcal{B}, \mathcal{O}$ are stored in computer memory, then in each particular case we need only to track up when the family $\{a_\lambda\}$ hits them.

Let us state the problem in the following form.

Prove that for any domain D in $\mathbb{R}^3$ or $\mathbb{R}^2$ the sets $\mathcal{D}(D), \mathcal{B}(D), \mathcal{O}(D)$ are nonempty.

So far this result is known only for rotationally symmetric domains in $\mathbb{R}^3$ [11]-[13]. Certainly, when it will be proved that the sets $\mathcal{D}(D), \mathcal{B}(D), \mathcal{O}(D)$ are non-empty (which is the main property of any set), the questions on the structure of these sets will arise. Each of them are likely to be stratified with respect to the codimensions of the bifurcations arising when the family $\{a_\lambda\}$ intersects them.

It would be natural to raise quite similar questions for periodic regimes (for instance, of fixed period p). Besides the sets $\mathcal{D}_p, \mathcal{B}_p, \mathcal{O}_p$, which naturally generalize the sets $\mathcal{D}, \mathcal{B}, \mathcal{O}$, we need to include here the *duplicator* $\mathcal{D}b_p$, i. e. the set of all time-dependent solenoidal vector fields in the domain D that are p-periodic in t and the corresponding monodromy operator has the multiplier -1. The nonlinear perturbations of this critical situation lead generically to the *period doubling bifurcation.*

Finally we notice that in the finite-dimensional case the definitions above work fairly well. For example, it turns out to be possible to construct sets similar to $\mathcal{D}, \mathcal{B}, \mathcal{O}$ for the Galerkin approximations to the Navier-Stokes equations (Yudovich V. I., On the bifurcators, oscillators and destabilizers of the Navier-Stokes system. (in preparation)).

Problem 4. *Completeness of the Floquet solutions system in the stability problem for periodic flows of viscous fluids.*

Linearization of Navier-Stokes equations, on a known $T-$periodic flow $a(x,t)$ in the (bounded) domain D with a rigid boundary, produces a system of equations with T-periodic coefficients. Searching for solutions of the form $e^{\sigma t}u(x,t)$, where the vector-function u is T-periodic, we obtain a spectral problem with the complex parameter σ:

$$\frac{\partial u}{\partial t} + \sigma u + (u, \nabla)a + (a, \nabla)u = -\nabla q + \nu \Delta u, \tag{14}$$

$$\nabla \cdot u = 0, \tag{15}$$

$$u|_{\partial D} = 0. \tag{16}$$

The set of complex numbers σ for which this problem has a nonzero solution is called *the stability spectrum* or *Floquet spectrum* of the flow $a(x, t)$. Let us note that if this spectrum contains a point σ, then it also contains a countable number of points $\sigma + in\omega$, $\omega = 2\pi/T$, $n \in Z$. Along with $e^{\sigma t}u(x, t)$, solutions of the form $e^{\sigma t} \sum_{k=0}^{n} t^k u_m^k(x, t)$, $m = 0, 1, \dots r$ are also referred to as Floquet solutions. Here u_m^k are T-periodic vector-functions, which are called adjoint Floquet solutions, or generalized Floquet solutions (which is extremely ambiguous).

Let us define the Hilbert space $S_2(D)$ as the closure of the set of all C^∞-smooth and compactly supported solenoidal vector fields on domain D in $L_2(D)$-norm. We formulate the following problem:

Prove that for any T-periodic solution a, the system of Floquet solutions is complete, i. e. their values at $t = 0$ form a complete system in $S_2(D)$. Let us introduce the *monodromy operator* U_T of the linearized system, which is obtained from (14)-(16) when $\sigma = 0$. By definition, for any solution $u(x, t)$ of this system we have $U_T u_0 = u(\cdot, T)$, where u_0 is the initial value: $u_0(x) = u(x, 0)$. An equivalent statement of the problem is:

Prove that the monodromy operator U_T has a complete system of eigen- and adjoint vectors.

In the case of a stationary flow, the completeness of the normal modes system was proved long ago by S.G.Krein (see [14]). The application of the well-known Keldysh theorem plays the crucial role in this proof. The idea is that for $a = 0$ we have the self-adjoint spectral problem, for which the completeness of the eigenvector system can be derived in a standard way, with the help of the Hilbert-Schmidt theorem. The terms that contain the flow a form though not small, but in a sense a weak perturbation of the basic self-adjoint positive-definite operator. This is what makes it possible to apply the Keldysh theorem.

It is rather surprising that in the periodic case the Keldysh theorem is non-applicable. Some results on the completeness of Floquet solutions were obtained in the 1970's by my PhD student A. I. Miloslavsky [15]. He instead applied the Dunford-Schwartz theorem [16], which also tells us that completeness of the root vector system is preserved under perturbation, though it is based on quite different principles than the Keldysh theorem. The conditions of this theorem prohibit the eigenvalues of the unperturbed operator to come arbitrarily close to each other. This kind of behavior is typical for the spectral boundary-value problems on the interval or on a plane domain. However for the Laplace operator (and for the Stokes operator as well) in a bounded domain D in R^m the eigenvalues λ_n grow like $n^{2/m}$ as $n \to \infty$ and for $m \geq 3$ approach each other. It is due to this restriction that the desired conclusion follows from Miloslavsky's general theorems only in the cases of total separation of variables when the problem is reduced to the second order parabolic equation (or system of such equations) with the coefficients periodic in t and only one spatial variable. This class certainly includes a lot of interesting flows, such as parallel flows in a circular pipe or in a channel, time-periodic symmetric flows between two coaxial cylinders, etc.. But the general problem turned out to be very complicated, and the difficulties here are of a fundamental nature.

Concentrating our attention on that essential properties of the linearized Navier-Stokes system, which we are really able to deal with, we come to the following abstract statement of the problem.

Consider the following ordinary differential equation in the Hilbert space H:

$$\frac{du}{dt} + Au \ = \ B(t)u, \tag{17}$$

where A is a (constant) self-adjoint operator (similar to the Laplace operator $-\Delta$ or the Stokes operator $-\Pi\Delta$), and its inverse operator $A^{-1} = G$ (Green operator) is completely continuous. The operator-function $B(t)$ is t-periodic with period T; for any t it is subordinate to the operator A in the strong sense which means that the operator-functions $G^{1/2}B(t)$ and $B(t)G^{1/2}$ are bounded and continuous in t with respect to uniform operator topology (generated by the operator norm on H). More precisely, operator $B(t)$ can be unbounded and not defined everywhere, but its domain ($dom(B(t))$) should be everywhere dense and contain the image of the operator $G^{1/2}$. At the same time the above mentioned operator-functions must be continuously extendable up to the bounded and continuous operator-functions.

In the case B is constant, the completeness can be derived from the

Keldysh theorem. Seemingly this result can be extended to operators $B(t)$ periodic in t. However, Miloslavsky constructed an example of an equation of the form (17), with the bounded coefficient $B(t)$, for which the monodromy operator is quasi-nilpotent! This equation does not have Floquet solutions at all. Note that the importance of the completeness property of the normal modes or of the Floquet solutions in the stability/instability problem is strongly exaggerated by many authors. The fact that the equation (17) does not have Floquet solutions indicates its *overstability*: each of its solutions decays faster than any exponent as $t \to +\infty$, at least like $e^{-kt\ln t}$ for some $k > 0$, or even like e^{-kt^α} for some $\alpha > 1$.

The example of Miloslavsky is of a rather abstract nature. It would be very interesting to determine if this kind of overstability is possible for parabolic partial differential equations and for the linearized Navier-Stokes system. My guess is that this is not possible; it is more likely that overstability exists on some invariant subspace. The reason is that any pair of differential operators with variable coefficients, say, of orders m and n, "almost commute", their commutator "loses the order". This differential operator is of order less than $m+n$. On the other hand, for the commuting (in the natural sense) operators A and $B(t)$ the conclusion on the completeness certainly holds.

I would also like to refer to the paper [18], where an example is constructed of a parabolic equation of the form $\frac{\partial u}{\partial t} - \Delta u = q(x,t)u$ on the torus T^3 with the coefficients bounded with respect to x, t. In this example the equation has some solutions decaying like e^{-ct^2}, $c > 0$ as $t \to +\infty$. However, the issue remains open whether or not such a fast damping is possible when the function q is periodic in t?

There exists a class of equations (17) with the self-adjoint and strictly positive monodromy operator . These are equations for which the condition ([12, 13]) is satisfied:

$$B^*(-t) = B(t). \tag{18}$$

In this case the monodromy operator of equation (17) certainly has an orthonormal eigenbasis. Unfortunately, only a few rotational periodic fluid flows and periodic convective flows of a stratified fluid lead to the equations of the form (17) with the condition (18) satisfied.

5 Stability of ideal fluid flows.

Problem 5. *Justify the validity of linearization in the problem on instability of a stationary flow of an ideal incompressible fluid with respect to weak norms.*

Instability must be understood as a lack of Lyapunov stability. The definition of Lyapunov stability uses the norm on the function space of solenoidal vector fields, tangent to the boundary of the flow domain. The answer to the stability question depends crucially on the choice of this norm [14, 19, 20, 21, 22]. There exist strong reasons to believe that all the flows of ideal incompressible fluid are unstable with respect to "strong" norms, such as $\max_x |\nabla \times v(x,t)| + \dots$ - in $3D-$case and $\max_x |\nabla(\nabla \times v(x,t))| + \dots$ — in $2D$-case. Here the dots stand for weaker norms, for instance L_2-norm. Although this statement in its general form still remains a hypothesis, a large number of examples and theorems, relating to various types of flow, strongly confirms its correctness. Apparently, even solid rotation of a fluid is not an exception.

Everything convinces us that even L_p-norms of a curl in the $3D$-case and its C^λ-norms in the $2D$-case also grow infinitely as $t \to \infty$ for very wide classes of flows. These classes are likely to be so wide that no stationary flow exists that is Lyapunov stable with respect to these norms.

In $2D$-case the known global existence theorem suggests the natural choice of the norm. It is the norm in the space V of solenoidal vector fields in the domain $D \subset R^2$ with a bounded vortex:

$$\| v \|_V = \operatorname{ess} \max_{x \in D} | \nabla \operatorname{curl} v(x) | + \dots \tag{19}$$

Here again dots denote a minor norm. The norm of the vector field $v(x,t)$ in the space V is estimated uniformly in $t \in R$ [26]. This is the strongest norm that is still uniformly bounded for all $t \in R$.

In any case, it is obvious that the stability and instability definitions are of interest only when stable flows exist. It is easy to check that flows with constant vortex are Lyapunov stable in the space V.

In the $3D$-case the "right" choice of norm is obscure, at least because we do not know of any global existence theorem for the initial boundary-value problem. At the same time, in the case of Lyapunov stability, according to the definition, the motion in the presence of small initial perturbations must be defined for any $t > 0$. That is true, in principle this does not prevent us from treating the collapse (or going of the motion to infinity for a finite time) as the special case of *instability*. May be we'll have to soften the definition of the Lyapunov stability admitting only smooth perturbations from some set which should be everywhere dense in the chosen function space. Somehow or other, for the moment there are only two reasonable candidates for the role of the "right" norm, namely C-norm and L_2-norm. In deciding of the problem 5 another choice is of course possible, however, the norm must be weaker than $\max|\operatorname{curl} v|$ in the case of 3-D flows.

At present only one general result on the justification of linearization in the stability problem for stationary flows of ideal incompressible fluid is known [23]. However, in this work instability, for the case when the stability spectrum contains a point of the right half-plane, was proved only for norms which were much too strong. The other weakness of this article is the presence of the additional restriction on the spectrum. This is the requirement for the spectrum to contain a spectral set which lies entirely in the right half-plane (or in fact a stronger requirement may be needed). I suppose this disadvantage can be easily removed with the use of M. G. Krein's [24] approach connected with so called "almost eigenvectors".

Problem 6. *Justification of the Arnold's method in the stability problem for an ideal fluid flow.*

In spite of the significant progress achieved with the application of the V. I. Arnold's method, beginning from his pioneering works of the middle of sixties (see [25]), many fundamental questions of the theory remained in a shadow and are still unclear.

Problem 6a. *Prove the Lyapunov stability in the space V in the case when the stationary flow satisfies the Arnold's criterion.*

Let me remind that in the case of a stationary flow with the stream function ψ which satisfies the equation

$$\psi = F(\Delta\psi), \tag{20}$$

this criterion requires that the quadratic form

$$\mathcal{H} = \mathcal{H}[\varphi] = \frac{1}{2}\int\limits_D \left[(\nabla\varphi)^2 + \frac{\nabla\psi}{\nabla\Delta\psi}(\Delta\varphi)^2\right] dxdy, \tag{21}$$

$$\varphi\big|_{\partial D} = 0. \tag{22}$$

be positive definite or negative definite.

With natural restrictions, V. I. Arnold proved the *a priori* estimate for L_2-norm of the vorticity perturbation $\|\Delta\varphi(\cdot,t)\|_{L_2(D)}$. However, for initial data of such kind, although we know the global existence theorem [26], there is no uniqueness theorem. For uniqueness it is sufficient to assume that the initial velocity belongs to V, i. e. $\Delta\varphi \in L_\infty(D)$ (see also [27], where the uniqueness is proved for some class of flows with unbounded vorticity). So the stability in this space is proved only in some impaired sense: even if there are many perturbed flows (corresponding to the same initial data) all of them are very close to the basic stationary flow. In a natural way, the following problem arises.

Problem 6b. *Prove (or disprove) the uniqueness of the solution to the basic initial boundary value problem for the Euler equations in a bounded domain D for the case when the initial vorticity belongs to $L_p(D)$ for some $p > 1$.*

So far we have to acknowledge that the Lyapunov stability in the space V is proved completely only for the flows with constant vorticity. By the way, for such kind of flows the form (21) is not defined and the result about stability is obtained directly. It is time to note that by no means all stationary flows satisfy the equation of the form (20) with a univalent and smooth function F. Generally, a stationary flow is defined by the equation

$$\frac{\mathcal{D}(\psi,\Delta\psi)}{\mathcal{D}(x,y)} = 0, \tag{23}$$

i. e the requirement that ψ and $\Delta\psi$ were functionally dependent. For instance, the starting hypotheses of Arnold are broken for the stream function defined uniquely by means of the boundary value problem

$$\begin{aligned} -\Delta\psi &= -\psi^3 + 1, \\ \psi\big|_{\partial D} &= 0. \end{aligned} \tag{24}$$

In this case $\psi = \sqrt[3]{\Delta\psi + 1}$, so the function F exists but it is not smooth. Another example in which generally there doesn't exist any univalent function F: the function ψ satisfies the equation

$$\psi^2 + (\Delta\psi)^2 = 1 \tag{25}$$

and the boundary condition $\psi\big|_{\partial D} = 0.0$ The stability problem for such kind flows remains completely uninvestigated.

Problem 6c. *Investigate the stability of the flows (24), (25)* and similar. It is most likely that all flows which do not admit a univalent function F are unstable. Thereupon I would like to draw attention to the works [28, 29] where it was proved that the solution of the problem about maximum of the kinetic energy on the set of isovortical vector fields leads to the flows with univalent dependence between ψ and $\Delta\psi$.

Further, it is important in principle to develop the Arnold approach, which is a special form of the direct Lyapunov method, as applied to the instability problem.

Problem 6d. *Prove the instability of a stationary flow in the case when the Arnold criterion is roughly violated.*

Seemingly, everything confirms the opinion of the famous author that his criterion is "close to necessary" (see, for instance, [30]). However, as a matter of fact, in hydrodynamics it is very rare for instability to be established using the direct Lyapunov method. A kind of exception is given by the results of V. Vladimirov [31] obtained with the use of virials in the problems relating to the motion of a body in a fluid.

It is a pity that the beautiful criterion for the stability of a three-dimensional stationary flow obtained by Arnold, as it was expected by its author, turned to be inapplicable to any flow excepting, may be, the rigid rotation.

Problem 6e *Does a stable three-dimensional stationary flow of an ideal incompressible fluid exist?*

It is likely that even rigid rotation is unstable with respect to strong norms, for instance with respect to the vorticity norm $\max|\operatorname{curl} v| + \ldots$. Thus, if the stable flow does exist we still need to explain in what sense (in what function space, etc) it is stable.

6 Stability of the simplest laminar flows and first transition.

Problem 7. *Prove that the Hagen-Poiseuille flow in a circular pipe as well as the Couette flow in a channel are absolutely stable (i.e. stable at any Reynolds number).*

This time we are dealing with the strictly formulated spectral boundary value problems for ordinary differential equations (see, for instance, [32]). In the case of the Poiseuille flow, confining ourselves to axisymmetric perturbations, we should prove that all eigenvalues σ of the following spectral problem are located in the left half-plane $\mathbb{R}\sigma < 0$:

$$\{(L-\alpha^2) - [\sigma + i\alpha R(1-r^2)]\}(L-\alpha^2)\psi = 0, \tag{26}$$
$$\psi = \psi' = 0, \quad r = 1, \tag{27}$$

where α is the wave number of the perturbation, R is the Reynolds number, σ is a complex parameter and the function $\psi = \psi(r)$ is defined on the segment $[0,1]$. The second order differential operator L is defined by the equality

$$L = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2} = \frac{d}{dr}\left(\frac{d}{dr} + \frac{1}{r}\right) \tag{28}$$

There is no boundary condition at $r = 0$ and it is easy to understand why: the points of the axis of the pipe are interior for the initial problem in the cylinder and there are no singularities at them. Instead of a boundary condition we state the "boundedness condition" coming from the requirement that the rate of energy dissipation is finite (or that the velocity field belongs to the class $W_2^{(1)}$). This condition has the form

$$\int_0^1 \left(|(L-\alpha^2)\psi|^2 + |\psi|^2\right) r dr < \infty. \tag{29}$$

In fact we must also handle the corresponding spectral boundary value problems for non-axisymmetric perturbations. They are well-known (see [32])and let me omit them here.

It is necessary to prove that for arbitrary real α and R all possible eigenvalues σ of the spectral problem (26)-(28) are situated in the left half-plane $\mathbb{R}\sigma < 0$.

Subject to $R = 0$ (and arbitrary α) we have the self-adjoint boundary value problem and all eigenvalues σ are negative(the equilibrium state is of course asymptotically stable). Being guided by the results of the perturbation theory and simple estimates of the σ-spectrum, it is easy to prove that the eigenvalues cannot go to infinity for finite values of the Reynolds number R. Hence, the absolute stability property is equivalent to the non-existence of a critical values of the Reynolds number.

We call critical the value $R = R_*$ such that there exists at least one eigenvalue σ_0 in the imaginary axis. It is convenient to express it in the form $\sigma_0 = -i\alpha cR$ with c the unknown real constant (the phase velocity of the neutral perturbation). Thus, the absolute stability problem can be formulated in the following form.

Prove that for any real α and R there exists only the zero solution of the differential equation

$$(L - \alpha^2)^2\psi - \lambda g(r)(L - \alpha^2)\psi = 0 \tag{30}$$

with the conditions (27), (28). Here we put

$$g(r) = 1 - r^2 - c; \quad \lambda = i\alpha R. \tag{31}$$

Let us underline that only pure imaginary eigenvalues have physical sense. Meanwhile if, instead of the boundary conditions (27), one takes the conditions

$$\psi = L\psi = 0, \quad r = 1, \tag{32}$$

then one obtains the self-adjoint Sturm-Liouville boundary value problem for the function $\omega = (L - \alpha^2)\psi$. In this case all eigenvalues of the parameter λ are real. This proves the absolute stability in the case of "soft" boundary conditions (32).

The idea appears to watch the change of eigenvalues λ resulting from the change of boundary conditions (27) into the conditions (32) and prove that they remain real. If this were the case then the problem would be resolved. Alas, this is true only in the case $c \notin (0, 1)$. However for $c \in (0, 1)$ the boundary value problem (26), (28), (32) admits complex (unreal)eigenvalues together with the set of real eigenvalues [33]. The fact that the phase velocity c should lie in the interval of values of the velocity of the Poiseuille flow can be obtained more directly from the integral estimates ([33]).

In the case of the Couette flow in a channel we come by the same way to the following problem.

Prove the absolute stability of the plane Couette flow in a channel by establishing that the boundary value problem

$$(D^2 - \alpha^2)^2 u = i\alpha R(y - c)(D^2 - \alpha^2)u, \quad D = \frac{d}{dy}, \tag{33}$$

$$u = Du = 0 \quad (y = \mp 1) \tag{34}$$

on the segment $[0, 1]$ *has only zero solution for all real* α *and* R*.*

It should be noted that beyond a shadow of a doubt the Poiseuille flow in a pipe and the Couette flow in a channel are absolutely stable. This statement is supported by repeated and very bulky calculations (that is true, time and again "no doubt" statements turn out to be wrong). Moreover, for the Couette flow "almost analytic" proof was constructed [34]. The numerical part of this work was reduced to the checking of some, not so complicated, inequality for the Bessel function.

However, I would like to believe that it is possible to construct some beautiful algebraic-analytical proof. I imagine the general theorem which imply without embarrassment the absolute stability of both flows. This theorem cannot be very general because it should be based on the rather deep and special properties of the linearized Navier-Stokes equations. The point is that for parallel flows in non-circular tubes, which are quite similar to the Poiseuille flow, stability most likely can be lost already for finite values of the Reynolds number. It would be desirable (and I hope not so difficult)to prove this rigorously for the tubes with elongated rectangular and elliptic cross-sections.

It is also interesting to note that for the Poiseuille-Couette flow in a channel with the profile $\mathcal{U}(y) = ay + b(1 - y^2)$, according to calculations of several authors, absolute stability takes place not only for $b = 0$ (pure Couette) but also for sufficiently small values of the parameter $k = \left|\frac{b}{a}\right|$, say for $k < k_*$; for $a = 0$ we have the Poiseuille flow in a channel which is unstable for large R. A really good theory should also predict the value k_* separating absolutely stable and unstable flows. The role of a computer should be reduced to the calculation of the concrete values of the parameters $k_* = k_*(\alpha)$ and $R_* = R_*(\alpha)$ for $k > k_*(\alpha)$.

Several other absolutely stable flows are known. This is, for example, Couette flow in the case when only the outer cylinder is rotating. Another example is given by the Kolmogorov spatially periodic flow with sinusoidal profile in the case of a short longitudinal period. For these particular flows the absolute stability is proved, and for the former even global (nonlinear) stability takes place [36]. However, the problems on global nonlinear stability and the development of turbulence still remain topical. We discuss these problems below,in sections V and VI.

Exchange stabilities principle.

When a parameter, on which the basic regime depends, arrives at its critical value, generically, the following two basic variants can occur:either a pair of complex conjugate eigenvalues appear on the imaginary axis or the eigenvalue $\sigma_0 = 0$. The term *oscillatory instability* is attributed to the former case while with the latter we connect the term *monotonous instability.*

In the second case we also say that the monotonicity principle or *the exchange of stabilities principle* takes place. The former term remained from that (short) time when researchers believed that in viscous fluid dynamics instability is always monotonous.

In several cases it turn out to be possible to prove the monotonicity principle rigorously. I mention the free convection problem in which the result was achieved by the reduction to the spectral problem for the self-adjoint operator. For the spatially periodic Kolmogorov flow with the velocity profile $\mathcal{U} = \sin y$ the monotonicity principle was proved in [35] with the help of explicit analytical considerations based on the possibility to express the characteristic equation by means of the continued fractions. For several special steady and periodic in t rotational flows, the monotonicity principle was established in [12, 13]. Sometimes in order to justify the monotonicity principle we can apply the theorem on the positive leading eigenvalue of a positive linear operator (Perron-Frobenius-Ientch-Rutman-M. G. Krein).

However, in the most interesting hydrodynamical case of the Couette-Taylor flow between the *co-rotating* rigid cylinders the principle is still not justified. The following mathematical problem is not resolved.

Prove that for an arbitrary $\alpha \in \mathbb{R}$ the minimal critical Reynolds number R of the spectral boundary value problem

$$(L - \alpha^2)^2 u - \sigma(L - \alpha^2)u = 2\alpha^2 R\omega(r)v, \tag{35}$$

$$(L - \alpha^2)v - \sigma v = -\lambda g(r)u, \tag{36}$$

$$u = u' = v = 0 \quad (r = r_1,\, r_2) \tag{37}$$

corresponds to the eigenvalue $\sigma = 0$.

Here r_1, r_2 are the radii of the cylinders, $0 < r_1 < r_2$, α is the axial wave number, the functions ω and g are expressed through the basic Couette profile

$$v_0 = v_0(r) = Ar + \frac{B}{r} \tag{38}$$

by the equalities

$$\omega(r) = \frac{v_0(r)}{r}; \quad g(r) = -\left(\frac{dv_0}{dr} + \frac{v_0}{r}\right). \tag{39}$$

The constants A and B are defined by the boundary conditions $v_0(r_1) = \Omega_1 r_1$, $v_0(r_2) = \Omega_2 r_2$, where Ω_1 and Ω_2 are the angular velocities of the cylinders. At that we should assume that $\omega(r) \geqslant 0$ for $r \in [r_1, r_2]$) and the Synge condition for instability $A < 0$ holds (for $A \geqslant 0$ stability takes place for all R, see [32]).

Repeated many times, calculations and natural experiments show us beyond a shadow of a doubt that the monotonicity principle is true. In some particular cases (narrow gap $r_2 - r_1$, close angular velocities Ω_1 and Ω_2)) it was actually proved. However the rigorous proof for the general case is still absent. Let me repeat, the problem here is not the mathematical rigor *per se*, but it is necessary to understand the reasons of the phenomenon. Why self-oscillations don't appear at the first transition? When the desirable result will be achieved, no doubt, it will enable us to decide, together with this particular case, many other problems and first of all for the rotational flows with profiles different from the Couette one (38).

Note that after the spectral problem (35)–(37) we must consider also the spectral problems for non-rotationally symmetric modes depending on the polar angle through the multiplier $e^{im\theta}$.

Existence of the infinite sequence of the critical values $R_1(\alpha) < R_2(\alpha) < \ldots$, going to infinity, at $\sigma = 0$ (and even for $\sigma > 0$) was proved by reduction to the integral equation with an *oscillatory* (in the Gantmakher-Krein's sense) kernel [36]. Thus we'll have the right to consider the problem as completely resolved when it will be rigorously proved that, as $R < R_1$, the entire σ-spectrum (including its part corresponding to the non-symmetric modes, for $m \neq 0$) is located in the left half-plane.

In the case of counter-rotating cylinders existence of monotonous instability critical values corresponding to creation of the Taylor vortices was proved in [37]. In this case however the monotonicity principle is not always valid:for large angular velocities of the outer cylinder the first transition can be connected with the appearance of the unstable oscillatory mode which is not symmetric.

In Soviet times scientists often had to answer the question about the economical effect of their results. They had to spin answers out of thin air.

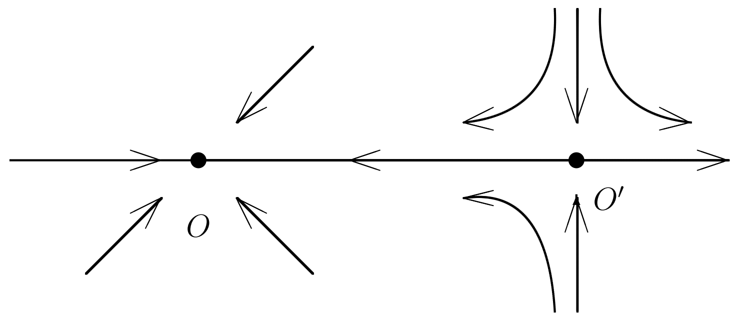

Figure 3:

However, in the case of the monotonicity principle this effect can be really evaluated in roubles or pounds. The point is that researchers time and again find themselves in a typical position described by Confucius. They have to catch a black cat in a dark room, and in addition the cat is absent. The huge work is necessary to get the result that oscillatory instability is impossible in this or that situation. And as the only result the melancholic sentence appears at the end of the paper: "Oscillatory instability is not found". Moreover, as it is impossible to get the absolute certainty, the following authors re-examine the result again and again. Only the rigorous proof of the monotonicity principle preserves us at one go from this Sisyphean toil and saves a lot of human and computer time.

Problem 9. Instability "in the large" of the Poiseuille flow in a pipe and the Couette flow in a channel (asymptotic bifurcation theory).

How to achieve the agreement between the result on the absolute stability of the Poiseuille flow in a circular pipe and the Couette flow in a channel and, on the other hand, the results of experimentalists who, beginning from Reynolds, report regularly about the observed instability and development of turbulent regimes? In general the frame of the answer is discernible though we are still far from the complete clarity. Most likely these flows, staying stable "in the small" for all Reynolds numbers, are globally stable only for sufficiently small R.However, beginning from some value of the Reynolds number R these flows become *unstable in the large.* The reason of this is that for $R \to \infty$ the domain of attraction is contracting to the point (at least in one direction) corresponding to the basic flow itself. Events of such kind happen, for instance, when some unstable equilibrium O' is coming closer

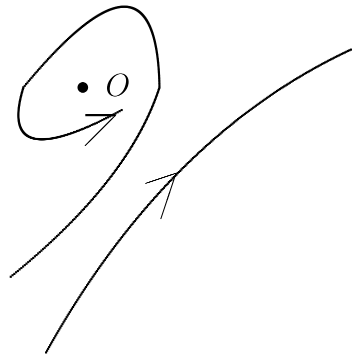

Figure 4:

and closer to the given asymptotically stable equilibrium O (see Fig. 3) and in the limit they merge together. Of course, it can happen that this is an unstable limit cycle or an even more complicated invariant set that approaches the equilibrium O. There is also another variant for which the equilibrium O is stuck in the limit into a separatrix trajectory (Fig. 4). The results of computer experiments seemingly testify to the first variant (Fig. 3) though the situation of Fig. 4 is more difficult to disclose and, may be, it will be also found out.

Problem 9a. *Prove that the Poiseuille flow in a pipe and the Couette flow in a channel are unstable in the large.*

It is very important to clarify the nature of this instability. If, indeed, the situation of Fig. 3 is realized, then the question is what is the nature of the unstable regime O'? The authors of the work [38] got over the huge computational difficulties and calculated the two-dimensional and three-dimensional soliton-like stationary regimes near the Couette flow in a channel (see also the references in this paper to the results of other authors on the existence nearby the Couette flow of space-periodic regimes even at lower Reynolds numbers). The results on intermittency of "turbulent plugs" and zones of the laminar Poiseuille flow in the transition interval of Reynolds numbers also can serve as some evidence on existence of soliton-like solutions to the Navier-Stokes equations.

Problem 9b. *Prove that for sufficiently large Reynolds numbers there exist stationary (plane and also periodic in the transversal direction)solutions* *to the Navier-Stokes equations tending to the Couette flow as* $|x| \to \infty$.

Problem 9c. *Prove that there exist the solutions of the Navier-Stokes system in a circular pipe of travelling soliton type* tending to the Poiseuille flow as $z - ct \to \infty$ (z is the axial variable, c is the phase velocity).

Problem 9d. *Prove the existence of space-periodic and soliton-like, stationary and periodic in time travelling waves tending, as* $R \to \infty$, *to the Poiseuille flow in a pipe and, respectively, to the Couette flow in a plane channel.*

It is most likely that all these problems 9a-9d will be resolved along with the construction of the *asymptotic bifurcation theory.* I mean the case when the critical Reynolds number is infinitely large: $R_* = \infty$. The fact is, of course, that $R = \infty$ is an essentially singular point for the Navier-Stokes system. Therefore such theory should include the construction of boundary layer (or may be other?) asymptotics of the secondary regimes merging with the basic one as $R \to \infty$.

When this theory is built (no doubt it will happen!), many other applications will be found in hydrodynamics as well as in other branches of mathematical physics.

7 Transitions and chaotic regimes.

Problem 10. *Find and rigorously justify the existence of strange attractors in the Navier-Stokes system and its nearest relatives* (convection problem, multi-component fluids, magnetic hydrodynamics, etc.)

In a series of hydrodynamical problems, with the use of natural and computer experiments, we have already settled the chains of transitions leading from an asymptotically stable equilibrium (stationary motion) through secondary equilibria, limit cycles and/or invariant two-dimensional tori to complicated chaotic regimes. The treatment of experimental data was obtained in the framework of ideas and images of the general bifurcations theory. However, even in the case of ordinary differential equations, for the most interesting cases this work was not carried through up to the strict checking of the conditions of general theorems. During the last decades, several prominent mathematicians even began to declare that in problems of such kind the abilities of rigorous mathematical analysis are exhausted and henceforward we have to pin our hopes only on computer calculations. And even the realization of rigorous proofs has been entrusted with computers. It is impossible to accept this point of view.

No doubt the direct numerical calculations always played a significant role in the development of mathematics. Archimedes, however, not even for a minute, thought it was sufficient to determine the volumes of a ball and a cone by weighing. He followed up on developing his methods of calculating volumes up to the triumphant conclusion. Euler, Gauss, Ramanujan got many of their discoveries, especially in number theory, as a result of extensive calculations and observations. However their discoveries became the property of mathematics only after the development of the corresponding rigorous theories. The application of computers widened the scope of numerical experiments utterly and now plays the decisive role in the investigation of the processes described by differential equations.

Perhaps mathematicians never thought that literally everything in the world must be justified with complete strictness. If we consider mathematics as a tool for investigating nature (for me this is only one but the main its side), then its characteristic feature is the aspiration to get absolutely reliable results. Meanwhile, very often it is sufficient to obtain results with probability 0.99 or maybe even 0.6. High reliability is very expensive–for instance, rigorous proof of the error estimates ("demonstrative calculations" according to K. I. Babenko) requires much more computer time than even the calculation itself.

It is not out of place to note that rigorous mathematical proofs give us results with complete reliability, but... in the limit $t \to \infty$ only. So many times it occurred that falseness of proofs for important results (or may be even of the results themselves) has been detected only years or even decades later. I heard that about 30% of theorems which are publishing in journals like "Comptes Rendus" or "Doklady Akademii Nauk SSSR" turn out to be wrong.

It seems obvious that the most fundamental results supporting a great deal in mathematics should be justified absolutely rigorously. Otherwise, very quickly, the house of cards effect will convert our reasoning into taking shots in the dark.

I admit that in future the rigorous justifications of results within computers such that their verification is also accessible for computers only will play the essential role, for example, in structural calculations in extremely important situations when human lives depend crucially on the work of the device. However, mathematics has its "human side". It is notable that recently this was an essential reason to increase the NSF grants for mathematicians.

Going back to our problem note that the highest chances to its resolution are connected with asymptotic methods and methods of the bifurcation theory. In particular, when investigating intersections of bifurcations in the Couette-Taylor problem (viscous flow between rigid rotating cylinders) the Navier-Stokes system is reduced to the *amplitude systems* on the central manifold ([39, 40]). I performed extensive computer experiments with these amplitude systems and found homoclinic bifurcations, the doubling cascades of limit cycles, resonance breaking up of tori and probably other transitions leading to the creation of various chaotic regimes. The original problem for the Navier-Stokes equations is reduced to the investigation of the system of ordinary differential equations of comparatively small order (sixth or eighth, and after the further reduction, even of fourth and respectively fifth order).

Now it is just the moment to recall that there are fairly big gaps even in the theory of ordinary differential equations. For instance S. Smale [1] states the problem which, in a somewhat free exposition, sounds as

"Prove that there exists the Lorenz attractor in the Lorenz system".

In fact, up to now the experimental observation of the Lorenz attractor and the general existence theorems are not united:nobody has examined the fulfilment of the general conditions for the special Lorenz system. I expect that the further development of asymptotic methods will enable us to resolve both this Smale's problem and our problem 10, say, for the Couette-Taylor problem. That is true, in the former case the analysis of the reduced system should be supplemented with the proof that the found attractors sustain the addition of (initially removed) higher order terms of the power series. This technical problem seems quite surmountable, though we should be ready to see that some of the observed attractors will disappear.

It is not so difficult to show other situations in fluid dynamics displaying different degenerated bifurcations and predisposed to appearance of strange attractors and chaotic regimes.

Of course, various limiting cases and first of all the case of vanishing viscosity suggest to us different ways for searching chaotic flow regimes.

8 Asymptotics of vanishing viscosity and turbulence.

It cannot be even doubted that the problem of fluid motion at very low viscosity (i.e. very large Reynolds numbers) is the central one in hydrodynamics. All the problems discussed above are its more or less essential parts.

Problem 11a. *Prove (or disprove) that for $\nu \to 0$ a solution to the Navier-Stokes system in a bounded domain $D \subset \mathbf{R}^m$ with a fixed rigid boundary ($v|_{\partial D} = 0$) and assigned initial velocity field ($v|_{t=0} = v_0(x)$) approaches the solution to the Euler equations with the same initial condition and the boundary condition ($v_n|_{\partial D} = 0$) .*

Assume that the data of the problem,namely $\partial D, v_0$, and exterior mass force (if it is present) are C^∞-smooth.

Of course, it should be specified, what kind of passage to the limit is in use here. Is this uniform convergence on an arbitrary interior subdomain? Or convergence in mean, in the norm $L_p(D)$? Or ,may be, in some measure?.. So far it is hard to state any reasonable conjecture regarding this matter.

The main difficulty here is related to the presence of a rigid wall. Even in the two-dimensional case, where we have strong existence theorems for the Navier-Stokes equations as well as for the Euler equations, the situation is completely unclear. If there is no boundary, say, in the case of spatially-periodic flows (equations on torus T^2), or there are "soft" boundary conditions: $v_n|_{\partial D} = 0,\ \nabla \times v|_{\partial D} = 0$, then everything is fine [41]: passage to the limit as $\nu \to 0$ is justified by using the integral estimates of vorticity.

The other complicated case is when the initial velocity field has a discontinuity on some curves. The case of *weak discontinuity*, when the velocity and hence the pressure are continuous and only the vorticity jump takes places, can be analyzed fairly well. The point is that in a fluid without a boundary, when the initial conditions are smooth, there are no boundary layers at all,and in the cases of a weak discontinuity or "soft" boundary conditions, boundary layer equations are linear and admit an explicit solution. It is also very important that we know in advance, where this boundary layer is forming - along the whole boundary (in the case of the soft boundary conditions) or along the whole weak discontinuity curve. On curves of strong discontinuity and on rigid walls, boundary layers are described by the Prandtl equations, which are nonlinear and cannot describe flow near the whole rigid boundary or the whole curve of strong discontinuity. The fundamental obstacle here is the *separation of a boundary layer.*

Let me notice that on a fluid free boundary the boundary layer is also weak and linear. (See [43,44] on the results of V.A.Batischev, V.V.Puhnachev, L.S.Srubschik). However, we are still far from the rigorous theory here because of the fundamental difficulties related to the existence theorems.

Problem 11b. *Determine the limit of a stationary solution to the Navier Stokes system as $\nu \to 0$. In particular, find asymptotics of the stationary flow of a viscous fluid past a rigid body.*

Besides difficulties related to separation of the boundary layer, in stationary (and periodic in t) problems we encounter the other serious difficulty in the determination of the limiting flow regime. The stationary Euler equations have infinitely many stationary solutions, and the question is which of them is the limiting one for the prescribed initial conditions. Things get even more complicated since we do not know beforehand the smoothness degree of this limiting stationary regime. If it is discontinuous, the location and nature of discontinuities also have to be determined.

Furthermore, the question on stability of these stationary regimes naturally arises. They are most likely unstable at large Reynolds numbers. However, fluid-dynamicists used to assume optimistically that the asymptotics is still valid for those moderate Reynolds numbers, at which stability is preserved. One can say in addition,that arising self-oscillatory regimes, at slightly supercritical Reynolds numbers, remain so close to the stationary flow, that integral characteristics, say the resistance force, differ very little from their stationary values.

Problem 11c. *Determine the average velocity field in developed turbulence as $\nu \to 0$ (Re $\to \infty$), and calculate correlations*

$$\langle v(x',t) \otimes v(x'',t) \rangle,$$

i.e. the average values of all possible products of the velocity field components at the prescribed time t in the points x' and x'' of the flow domain. Determine also correlation functions corresponding to the distinct times t' and t''

$$\langle v(x',t') \otimes v(x'',t'') \rangle.$$

A huge amount of literature is devoted to this problem [46] (see also recent review [47]). However, all existing turbulence theories without exception are based on hydrodynamics equations only to some rather small extent.

In any case, all of them include some hypotheses that are not derived from the Navier-Stokes equations, and even possibly contradict them. It is worth noticing that some of these theories predict the average velocity field quite well. For instance, it is hardly an accidental coincidence that the calculated in [49] average profile of turbulent Couette flow in a channel is the same as the one obtained experimentally. However,I suppose, no one theory manages to predict more complicated flow characteristics, beginning with second order correlation functions, not to mention the higher orders. I do not have a sufficiently clear answer to the inevitably arising question on the exact type of averaging that we should bear in mind while stating the problem 11c. The most common ones are averaging with respect to time and with respect to an invariant measure on a phase space of a system. In the case of ergodicity these two types of averaging are the same. While pressing for the theoretical results to coincide with the experimental data, we shall probably need to take into account that a measuring device makes some kind of spatial averaging over a small region.

However, against all the odds I believe that the consecutive theory of developed turbulence, which describes flows with very low viscosity, can be and will be built. It is not inconceivable that this theory will bifurcate into several branches. Probably, we will have to describe differently the turbulent flows in pipes, turbulent convective flows and Couette-Taylor flows, turbulent flow past a body etc. That is true, the lack of a general theory leads to much more branching. Almost every flow needs to be considered separately, introducing every time new *ad hoc* hypotheses. So far the problem is to construct the asymptotics for at least one particular case.

Sometimes experiments provide us with so beautiful and clear results that it is a shame on theorists that they cannot interpret them. For instance, for the Couette-Taylor flow between two cylinders (when the exterior cylinder is fixed) Taylor obtained (1923) the expression for the azimuthal velocity $v_\theta = \frac{c}{r}$, which is valid everywhere except narrow boundary layers (see [48]). It is quite remarkable that Taylor's vortices, which have lost their stability long ago, come alive once again for very large Reynolds numbers. Apparently, they survive (stay stable) for arbitrary large Reynolds numbers. An excellent review on turbulent Taylor vortices is presented in [45]. A large number of simple and explicit relations has been also discovered in experiments on Bénard's convection in a horizontal fluid layer. I believe that the problem of rigorous mathematical interpretation of these phenomena and patterns is not hopeless.

There is no reason to expect uniqueness in the problems 11b and 11c. It is quite possible that there exist several stationary regimes having different asymptotical behavior at vanishing viscosity. In fact, this is the case for the Couette flow between two rigid spheres as well as for Karman's flow between rotating planes. There exist some other turbulent regimes along with Taylor's turbulent vortices,and moreover the former themselves are not uniquely defined for given boundary conditions (their axial wave numbers varies).

In conclusion of this section, I would like to emphasize that it was the developed turbulence,at the limit of vanishing viscosity,that we discussed here. Ideally, the transition theory for moderate Reynolds numbers must describe all possible types of flow regimes, the conditions of their births and deaths while moving along parameters, and also provide us with the methods to compute them. It seems impossible to predict without particular calculations, what sequence of transitions will be realized in a prescribed situation. The theory must teach researchers rather the rules, according to which flow regimes are replaced with others, and the methods for the calculation of various regimes. Of course, it is very important to learn how to formulate the right questions and to point out the quantities that need to be calculated first of all. One would say that this theory will more resemble traffic regulations than a train schedule. Although, I suppose, we have good chances that asymptotic methods will enable us to predict even the transition sequences in various limiting cases.

ACNOWLEDGMENTS. The research described in this publication was made possible in part by Award No.RMI-2084 of the U.S.Civilian Research & Development Foundation for the Independent States of the Former Soviet Union (CRDF).

I am sincerely grateful to Natalya Popova and Marc Burton for their invaluable help in translation of this paper into English.

I am greatful to the reviewer for the suggestion to include the interesting modern books [50, 51, 52] into the list of referencies. Meanwhile one should remember that nothing can substitute for reading of pioneers—such authors as Lyapunov, Leray, Hopf.

Let me conclude this paper with some small edification for a young researcher who intends to start solving one or another of the problems discussed here. All fore-quoted authors (and the present author is of course not an exclusion) have never resolved these fundamental problems. Therefore one should study their works with care not allowing to carry oneself along a hopeless way.

References

[1] S. Smale. Mathematical Problems for the Next Century // The Mathematical Intelligencer, vol. 20, no. 2, 1998, pp. 7–15.

[2] Martynov G. A. Phase transitions problem in statistical mechanics // Usp. phys. nauk (1999) Vol. 169. no. 6. pp. 595–624(Russian)

[3] Fridkin V. M. Ferroelectrics–semi-conductors. Moscow, Nauka, 1976. 408 pp. (Russian)

[4] Yudovich V. I. Global regularity vs collapse in dynamics of incompressible fluids // Editor V. B.Filippov. In “Mathematical events of XX century”. M.: “Fazis”. (To appear).

[5] Ladyzhenskaya O. A. The mathematical theory of viscous incompressible flow. New York - London - Paris: Gordon and Breach Science Publishers. XVIII, 224 p. (1969).)

[6] Finn R., Smith D. R. On the stationary solutions of Navie-Stokes equations in two dimensions. Arch. Rat. Mech. and Anal. **25**. No. 1 (1967). 26–39.

[7] Leray J. Etude de diverses équations integrales non lineaires et de quelques problèmes que pose l’hydrodynamique. J. Math. Pures Appl., serie 9, **12** (1933), 1–82.

[8] Yudovich V. I. Periodic motions of a viscous incompressible fluid // Sov. Math., Dokl. 1960, vol. 1, pp. 168-172.

[9] Yudovich V. I. Rotationally-symmetric flows of incompressible fluid through a circular ring. Part 1, 2 // Preprint VINITI. 2001.

[10] Hopf E. Ein allgemeiner Endlichkeitsatz der Hydrodynamik. Math. Ann. **117** (1941), 764–775.

[11] Yudovich V. I. An example of loss of stability and generation of a secondary flow in a closed vessel (Russian, English) Math. USSR, Sb. 3,519-533 (1967).

[12] Yudovich V. I. On periodic differential equations with self-adjoint monodromy operator //Doklady RAN, 1999, Vol. 368, no. 3, 338-341.

[13] Yudovich V. I. Periodic differential equations with self-adjoint monodromy operator.// Math. Sbornik. 2001. Vol. 192. no. 3. pp. 137–160 (Russian).

[14] Yudovich V. I. "The Linearization Method in Hydrodynamical Stability Theory", In.: Translation of mathematical monographs, **74**, American Mathematical Society, Providence, Rhode Island, 177 p., (1989))

[15] a) Miloslavsky A. I. The Floquet theory for abstract parabolic equations. PHD thesis,Rostov-on-Don, 1976.
b) Miloslavsky A. I. The Floquet theory for abstract parabolic equations// Func.Anal.Vol.10, No2 (1976), 80–81.
c) Miloslavsky A. I.The Floquet theory for abstract parabolic equations.Basis properties of the generalized eigen-spaces of the monodromy operator// Preprint VINITI, 1975, no. 3073-75.
d) Miloslavsky A. I. Floquet solutions and reducibility of a class of differential equations in a B-space//Izv.SKNC VS,Nat.sci. 4 (1975), 82–87.

[16] Dunford N., Schwartz J. T. Linear Operators. Part III. Spectral operators. New York-London-Sydney-Toronto: Wiley-Interscience, 1971)

[17] Hopf E. Über die Anfangswerwertaufgabe für die hydrodynamischen Grundgleichungen. Math. Nachrichten, **4** (1950–51), 213–149.

[18] Meshkov V. Z. On the possible rate of decay at infinity of solutions of second order partial differential equations. Math. Sbornik. 72, No.2, 343-361 (1992).

[19] Yudovich V. I. On the loss of smoothness for the Euler equations// "Dinamika sploshnoy sredy". 1974. Issue 16. (Russian).

[20] Yudovich V. I. On the gradual loss of smoothness and intrinsic instability of an ideal fluid flows // Dokl. Akad. Nauk Vol. 370. no. 6, 2000. C. 760-763

[21] Yudovich V. I. On the loss of smoothness of the solutions to the Euler equations and the inherent instability of an ideal fluid flows // Chaos. 2000. Vol. 10, Issue 3. pp. 705-719.

[22] Yudovich V. I. On the unbounded increase in vorticity and velocity circulation in stratified and homogeneous fluid flows. (Russian, English) Math. Notes 68, No.4, pp. 533-540 (2000).

[23] Friedlander S., Strauss W., Vishik M. Nonlinear instability in an ideal fluid. // Ann. Inst. Henri Poincare, Anal. Non Lineaire 14, No. 2, 187-209 (1997).

[24] Daletsky Yu. L., Krein M. G. The stability of solutions of differential equations in a Banach space. Moscow, 1970. 534 p.

[25] Arnold V. I. Sur la géométrie différentialle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits, Ann. Inst. Fourier **16** (1966), 316–361.

[26] Yudovich V. I. Plane unsteady motion of an ideal incompressible fluid // Sov. Phys., Dokl., 1961. Vol. 6, no. 3. Pp. 18-20.

[27] Yudovich V. I. Uniqueness theorem for the basic non-stationary problem in the dynamics of an ideal incompressible fluid // Mathematical research Letters, 2, 1995. Pp. 27-38.

[28] Shnirelman A. Lattice theory and flows of ideal incompressible fluid, Russ. J. Math. Phys. **1** (1993), no. 1, 79–105)

[29] Burton G. R. Rearrangements of functions, maximizations of convex functional, vortex rings, Math. Annalen **276** (1987), 225–253. Variational problems on classes of rearrangements and multiple configurations for steady vortices, Ann. Inst. Henry Poicaé **6** (1989), no. 4, 295–319.

[30] Belenkaya L, Friedlander S., Yudovich V. I. The unstable spectrum of oscillating shear flows // SIAM Journal of Applied Mathematics. 1999, Vol. 59, no. 5, pp. 1701-1715.

[31] Vladimirov V. A. On the instability of equilibrium of fluids. J. Appl. Mech. Tech. Phys. **30** (1089), no. 2, 269–276. Direct Lyapunov method in problems of fluid equilibrium instability, Arch. Mech. **42** (1990), no. 4–5, 595–607.

[32] Lin C. C. The theory of hydrodynamic stability. Cambridge at the University Press, 1955.

[33] Yudovich V. I. On some spectral problems with differential weight (towards the absolute stability problem for the Poiseuille flow) // Preprint VINITI 1988, no. 2168-V88.

[34] Romanov V. A. Stability of plane-parallel Couette flow — Institute of Mechanics Problems AN USSR, 1971, preprint no. 1, pp. 26.

[35] Meshalkin L. D., Sinay Ya. G. Investigation of the stability of a stationary solution of a system of equations for the plane movement of an incompressible viscous liquid // translation from Prikl. Mat. Mekh. 25, 1140-1143 (1961))

[36] Yudovich V. I. Secondary flows and fluid instability between rotating cylinders (Russian, English) PMM, J. Appl. Math. Mech. 30, 822-833 (1966)

[37] Barkovsky Yu. S., Yudovich V. I. Spectral properties of a class of boundary value problem// Math. USSR Sbornik, Vol. 42 (1982), No. 3. 387-398.

[38] Cherhabili A., Ehrenstein U. Finite-amplitude equilibrium states in plane Couette flow // J. Fluid Mech. (1997), vol. 342, pp. 159–177.

[39] Chossat P., Iooss G. The Couette-Taylor Problem. Applied Mathematical Sciences. Vol. 10. Springer-Verlag New York, 1994.

[40] Kolesov V. V., Yudovich V. I. Calculation of oscillatory regimes in Couette flow in the neighborhood of the point of intersection of bifurcations initiating Taylor vortices and azimuthal waves. // Fluid Dyn. 33, No.4, 532-542 (1998).

[41] Yudovich V. I. Non-stationary flow of an ideal incompressible liquid// U.S.S.R. Comput. Math. Math. Phys., vol. 3, pp. 1407-1456. (1963)

[42] Srubshchik L. I., Yudovich V. I. Asymptotic behavior of first-order discontinuities of fluid flows with vanishing viscosity // Sov. Phys., Dokl., vol. 16, pp. 530-533, (1971)

[43] Batishchev, V.A.; Srubshchik, L.S. Diffusion of a spherical Hill vortex with vanishing viscosity. (Russian, English) Sov. Phys., Dokl. 16, 286-288 (1971).

[44] Pukhnachev V. V. Nonclassical problems of the theory of the boundary layer. Ministerstvo Vysshego i Srednego Spetsial'nogo Obrazovaniya RSFSR, Novosibirsk State University,Novosibirsk, 75 p. 1979.

[45] Koschmieder E. L., Bénard Cells and Taylor Vortices. Cambridge University Press, 1993. 337 p.

[46] Monin A. S., Yaglom A. M. Statistical Fluid Mechanics, The Mechanics of Turbulence, Vol. 1, (1971) MIT Press, Cambridge, Mass.; Statistical Fluid Mechanics, The Mechanics of Turbulence (revised and augmented edition), Vol. I, Chap. 2, CTR Monograph, Center for Turbulence Research, Stanford.)

[47] Yaglom A. M. New Trends in Turbulence, Proc. of Summer School, Les Houches (to be published)

[48] Townsend A. A. The structure of turbulent shear flow. Cambridge University Press, 1956.

[49] Berdichevsky V. L., Fridlyand A. A., Sutyrin V. G. Prediction of turbulent velocity profile in Couette and Poiseuille flows from the first principles // Phys. Rev. Lett., vol. 76, no. 31, pp. 3967–3970, 1996.

[50] Constantin P., Foias C. Navier-Stokes Equations. University of Chicago Press, 1988.

[51] Gallavotti G. Foundations of Fluid Dynamics. Springer-Verlag, 2001.

[52] Chemin J.-Y. Perfect Incompressible Fluid. OUP, 1998.